Nonequilibrium dynamics

of scalar fields

in a thermal bath

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Out-of-equilibrium processes in the early Universe

1. Formation of light elements/BBN

2. Decoupling of the photons =>CMBR

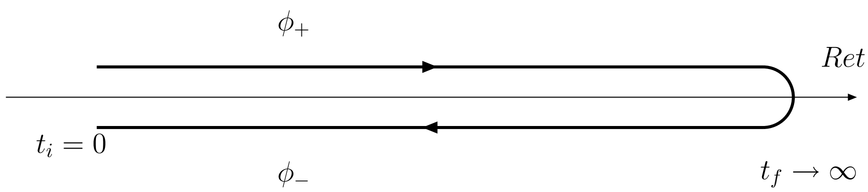
3. Preheating after inflation

4. Dark matter production

5. Matter/antimatter production

Sometimes the canonical treatment by Boltzmann's eqns. is possible, but, in general, full quantum treatment is plausible

Formalism



Keldyish contour

$$\Delta_C(x_1, x_2) =$$

$$\theta_C(t_1 - t_2)\Delta^>(x_1, x_2) + \theta_C(t_2 - t_1)\Delta^<(x_1, x_2)$$

Here the θ-functions enforce path ordering along the contour C, with $t_j = \mathrm{Re}(x_j^0)$, and

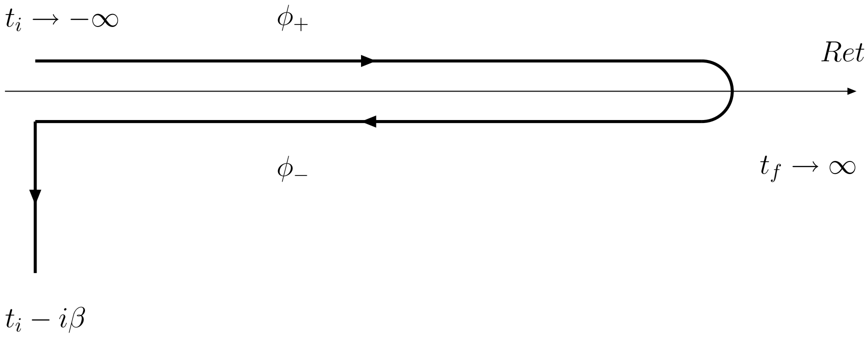
$$\Delta^>(x_1, x_2) = \langle \Phi(x_1)\Phi(x_2) \rangle = \langle \Phi^+\Phi^- \rangle$$
$$\Delta^<(x_1, x_2) = \langle \Phi(x_2)\Phi(x_1) \rangle = \langle \Phi^-\Phi^+ \rangle$$

- The Dyson-Schwinger eqns. on a Keldyish contour

$$(\Box_1 + m^2)\Delta_C(x_1, x_2) +$$

$$\int_C d^4x' \Pi_C(x_1, x')\Delta_C(x', x_2) = -i\delta_C(x_1 - x_2)$$

- Π represents the thermal bath (invariant under the time translations)



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Path in the complex time plane for thermal Green functions.

$$\Delta_{-+}(x_1,x_2) = \Delta^{>}(x_1,x_2), \quad \Delta_{+-}(x_1,x_2) = \Delta^{<}(x_1,x_2)$$

$$\Pi_{-+}(x_1,x_2) = \Pi^{>}(x_1,x_2), \quad \Pi_{+-}(x_1,x_2) = \Pi^{<}(x_1,x_2)$$

- Δ_{++}, Π_{++} and Δ_{--}, Π_{--} are causal and anti-causal Green functions

Kadanoff-Baym equations:

$$(\Box_1 + m^2)\Delta^{<}(x_1,x_2) =$$

$$\int d^4x' \, (\Pi_{++}(x_1,x')\Delta^{<}(x',x_2) + \Pi^{<}(x_1,x')\Delta_{--}(x',x_2))$$

$$(\Box_1 + m^2)\Delta^{>}(x_1,x_2) =$$

$$\int d^4x' \, (\Pi^{>}(x_1,x')\Delta_{++}(x',x_2) + \Pi_{--}(x_1,x')\Delta^{>}(x',x_2))$$

It is convenient to define the following functions:

$$\begin{aligned}
\Delta^R(x_1,x_2) &= \theta(t_1-t_2)(\Delta^>(x_1,x_2)-\Delta^<(x_1,x_2)) \\
&= \theta(t_1-t_2)\langle[\phi(x_1),\phi(x_2)]\rangle \\
&= \Delta_{++}-\Delta_{+-}=\Delta_{-+}-\Delta_{--} \\
\Delta^A(x_1,x_2) &= -\theta(t_2-t_1)(\Delta^>(x_1,x_2)-\Delta^<(x_1,x_2)) \\
&= -\theta(t_2-t_1)\langle[\phi(x_1),\phi(x_2)]\rangle \\
&= \Delta_{++}-\Delta_{-+}=\Delta_{+-}-\Delta_{--} \\
\Pi^R(x_1,x_2) &= \theta(t_1-t_2)(\Pi^>(x_1,x_2)-\Pi^<(x_1,x_2)) \\
&= \Pi_{++}-\Pi_{+-}=\Pi_{-+}-\Pi_{--} \\
\Pi^A(x_1,x_2) &= -\theta(t_2-t_1)(\Pi^>(x_1,x_2)-\Pi^<(x_1,x_2)) \\
&= \Pi_{++}-\Pi_{-+}=\Pi_{+-}-\Pi_{--}
\end{aligned}$$

$$\begin{aligned}
\Delta^-(x_1,x_2) &= i\langle[\phi(x_1),\phi(x_2)]\rangle \\
\Delta^+(x_1,x_2) &= \frac{1}{2}\langle\{\phi(x_1),\phi(x_2)\}\rangle
\end{aligned}$$

and

$$\begin{aligned}
\Pi^-(x_1,x_2) &= \big(\Pi^>(x_1,x_2)-\Pi^<(x_1,x_2)\big) \\
\Pi^+(x_1,x_2) &= -\frac{i}{2}\big(\Pi^>(x_1,x_2)+\Pi^<(x_1,x_2)\big)
\end{aligned}$$

Kadanoff-Baym eqns.

$$
\begin{aligned}
(\Box_1 + m^2)\Delta^-(x_1,x_2) &= -\int d^3\mathbf{x}' \int_{t_2}^{t_1} dt' \Pi^-(x_1,x')\Delta^-(x',x_2) \\
(\Box_1 + m^2)\Delta^+(x_1,x_2) &= -\int d^3\mathbf{x}' \int_{t_i}^{t_1} dt' \Pi^-(x_1,x')\Delta^+(x',x_2) \\
&\quad + \int d^3\mathbf{x}' \int_{t_i}^{t_2} dt' \Pi^+(x_1,x')\Delta^-(x',x_2)
\end{aligned}
$$

Spatial rotation invariance=>

$$
\begin{aligned}
(\Box_1 + m^2)\Delta^-(t_1,t_2) &= -\int_{t_2}^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1,t')\Delta^-_{\mathbf{q}}(t',t_2) \\
(\Box_1 + m^2)\Delta^+_{\mathbf{q}}(t_1,t_2) &= -\int_{t_i}^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1,t')\Delta^+(t',t_2) \\
&\quad + \int_{t_i}^{t_2} dt' \Pi^+_{\mathbf{q}}(t_1,t')\Delta^-_{\mathbf{q}}(t',t_2)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta^-_{\mathbf{q}}(t_1,t_2)|_{t_1=t_2} &= 0 \\
\frac{\partial}{\partial t_1}\Delta^-_{\mathbf{q}}(t_1,t_2)|_{t_1=t_2} = -\frac{\partial}{\partial t_2}\Delta^-_{\mathbf{q}}(t_1,t_2)|_{t_1=t_2} &= 1 \\
\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\Delta^-_{\mathbf{q}}(t_1,t_2)|_{t_1=t_2} &= 0
\end{aligned}
$$

Solution of the first KB eqn.

$$(\Box_1 + m^2)\Delta^-_{\mathbf{q}}(t_1, t_2) +$$

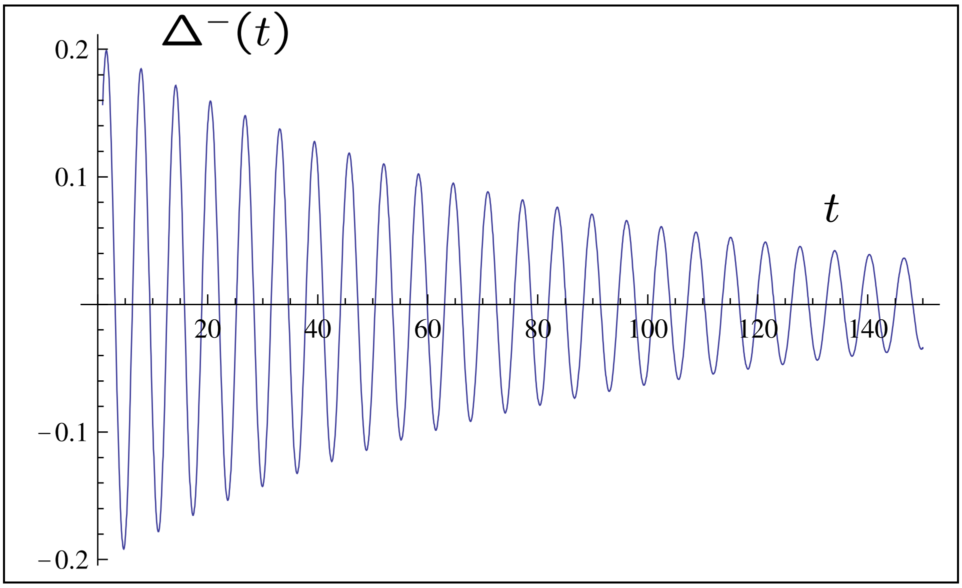
$$\int_{t_2}^{t_1} dt' (\Pi^-_{\mathbf{q}}(t_1, t'))\Delta^-_{\mathbf{q}}(t', t_2) = 0$$

The most general solution depends only the relative time difference:

$$\Delta^-_{\mathbf{q}}(t_1, t_2) = \Delta^-_{\mathbf{q}}(t_1 - t_2)!$$

A few steps of the proof:

- Start from the vacuum (time translational!) solution, i.e. coupling is set to zero
- Perturb to first order in coupling: again time translational solution
- Construct the solution to an arbitrary order in perturbation theory
- Strong coupling?: there is a general proof of time translational invariance (more involved)



$$(\Box + m^2)\Delta^-_{\mathbf{q}}(t) +$$

$$\int_0^t dt'(\Pi^-_{\mathbf{q}}(t-t'))\Delta^-_{\mathbf{q}}(t') = 0$$

Can be solved by applying the Laplace transform

The solution is:

$$\Delta^-_{\mathbf{q}}(t) = \int_{-\infty}^{\infty} \sin(\omega t)\rho_{\mathbf{q}}(\omega)d\omega$$

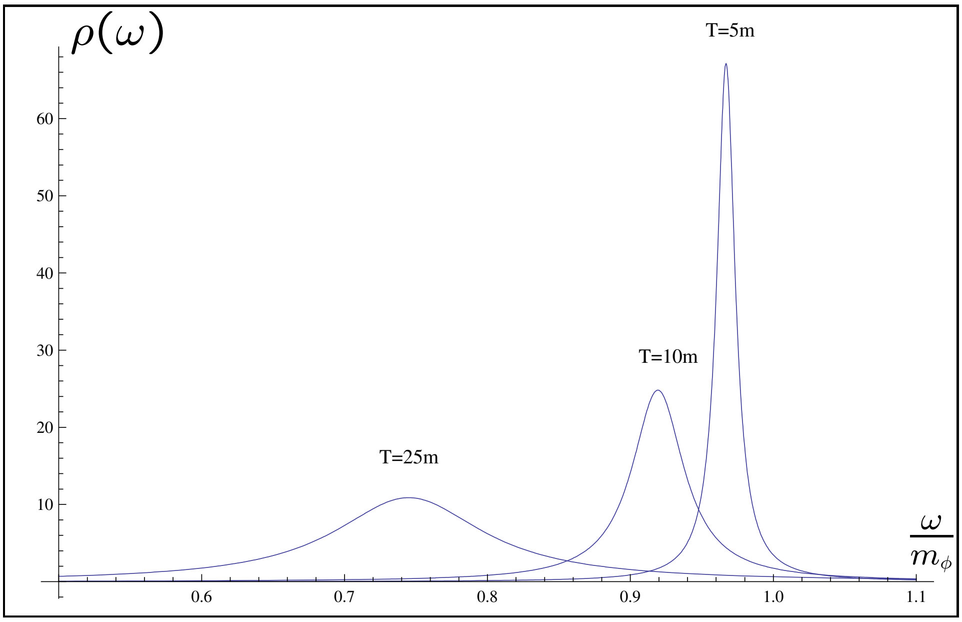
$\rho_{\mathbf{q}}(\omega)$ is a *spectral function*

$$\rho_{\mathbf{q}}(\omega) = \frac{-2\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) - 4\omega\epsilon}{[\omega^2 - \omega^2_{\mathbf{q}} - \mathrm{Re}\Pi^R_{\mathbf{q}}(\omega)]^2 + [\mathrm{Im}\Pi^R_{\mathbf{q}}(\omega) + 2\omega\epsilon]^2}$$

The spectral function at different temperatures

Reference model:

$$\mathcal{L} = \mathcal{L}_{\chi_{1,2}} + \mathcal{L}_{\Phi} + g\Phi\chi_1\chi_2$$



- The width is proportional to the $\text{Im}\Pi^R$

- The peak is located at the pole mass:

$$\omega_P^2 - \omega_{\mathbf{q}}^2 - \text{Re}\Pi_{\mathbf{q}}^R(\omega_P)]^2 = 0$$

i.e. gives the dispersion relation and the decay width of the *quasiparticle*

Solution of the second KB eqn.

$$\begin{aligned}\Delta^+_{\mathbf{q}}(t_1,t_2) &= -\int_0^{t_1} dt' \Pi^-_{\mathbf{q}}(t_1,t')\Delta^+_{\mathbf{q}}(t',t_2) \\ &\quad + \int_0^{t_2} dt' \Pi^+_{\mathbf{q}}(t_1,t')\Delta^-_{\mathbf{q}}(t',t_2)\end{aligned}$$

Define

$$\xi_{\mathbf{q}}(t_1,t_2) \equiv \int_0^{t_2} dt' \Pi^+_{\mathbf{q}}(t_1,t')\Delta^-_{\mathbf{q}}(t',t_2)$$

The solution which removes the r.h.s. is of the form

$$\int_0^{t_1} dt' \Delta^-_{\mathbf{q}}(t_1-t')\xi(t',t_2),$$

Above solution is usually referred as *memory integral*. After some algebra one has the *memory integral* written as

$$-\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Pi^+_{\mathbf{q}}(\omega)\mathcal{H}_{\mathbf{q}}(\omega,t_1)\mathcal{H}^*_{\mathbf{q}}(\omega,t_2)e^{i\omega(t_1-t_2)}$$

where

$$\mathcal{H}_{\mathbf{q}}(\omega,t) = \int_0^t d\tau e^{-i\omega\tau}\Delta^-_{\mathbf{q}}(\tau)$$

- Full solution has the form:

$$\begin{aligned}
\Delta^{+}_{\mathbf{q}}(t_1,t_2) &= a_{\mathbf{q}}\dot{\Delta}^{-}_{\mathbf{q}}(t_1)\dot{\Delta}^{-}_{\mathbf{q}}(t_2) \\
&+ b_{\mathbf{q}}\left(\dot{\Delta}^{-}_{\mathbf{q}}(t_1)\Delta^{-}_{\mathbf{q}}(t_2)+\Delta^{-}_{\mathbf{q}}(t_1)\dot{\Delta}^{-}_{\mathbf{q}}(t_2)\right) \\
&+ c_{\mathbf{q}}\Delta^{-}_{\mathbf{q}}(t_1)\Delta^{-}_{\mathbf{q}}(t_2) \\
&- \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Pi^{+}_{\mathbf{q}}(\omega)\mathcal{H}_{\mathbf{q}}(\omega,t_1)\mathcal{H}^{*}_{\mathbf{q}}(\omega,t_2)e^{i\omega(t_1-t_2)}
\end{aligned}$$

- Only the *memory integral* effects the late time behavior. In the limit $t_{1,2}\to\infty$ one has

$$\Delta^{+}_{\mathbf{q}}(t_1,t_2)=\Delta^{+}_{\mathbf{q}}(t)=-\frac{i}{2}\int_{-\infty}^{\infty}d\omega\,\cos(\omega t)\coth\left(\frac{\beta\omega}{2}\right)\Delta^{-}_{\mathbf{q}}(\omega)$$

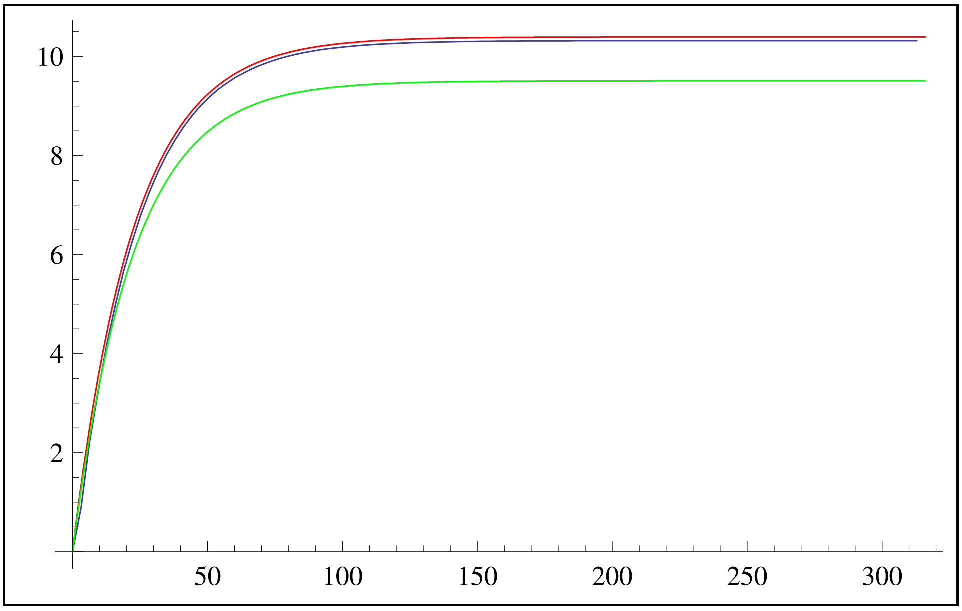
- Late time behavior is as expected. KMS condition:

$$\Delta^{+}_{\mathbf{q}}(\omega)=-\frac{i}{2}\coth\left(\frac{\beta\omega}{2}\right)\Delta^{-}_{\mathbf{q}}(\omega)$$

- Constants a,b,c set initial conditions. E.g. "tsunami" type:

$$\begin{aligned}
a_{\mathbf{q}} &= \frac{1}{2\omega_P}(1+2n_q) \\
b_{\mathbf{q}} &= 0 \\
c_{\mathbf{q}} &= \frac{\omega_P}{2}(1+2n_q)
\end{aligned} \quad (1)$$

Comparison with the Boltzmann's eqns.



Define the number density operator:

$$\hat{N}_q(t) = \frac{1}{2\omega_P}\left(\hat{\dot{\Phi}}_{\mathbf{q}}(t_1)\hat{\dot{\Phi}}_{-\mathbf{q}}(t_2) + \omega_P^2 \hat{\Phi}_{\mathbf{q}}(t_1)\hat{\Phi}_{-\mathbf{q}}(t_2)\right)\Big|_{t_1,t_2=t} - \frac{1}{2}$$

or

$$\langle \hat{N}_{\mathbf{q}}(t)\rangle = \frac{1}{2\omega_P}\left(\partial_{t_1}\partial_{t_2} + \omega_P^2\right)\Delta_{\mathbf{q}}^{+}(t_1,t_2)\big|_{t_1,t_2=t} - \frac{1}{2}$$

Since the solution is known=>

$$\langle \hat{N}_{\mathbf{q}}(t)\rangle\big|_{t\to\infty} = n_b(\omega_P)\left(1 + \mathcal{O}\left(\frac{\mathrm{Im}\Pi^{\mathrm{R}}}{\omega_{\mathrm{P}}^2}\right)^2\right)$$

The occupation numbers are in accord with the Bose-Einstein statistics but for *quasiparticles* with the spectrum defined by the *spectral function* $\rho_{\mathbf{q}}(\omega)$

Conclusions

- Full treatment is possible if the backreaction from the thermal bath can be neglected
- Potentially significant deviation from the solution of the Boltzmann's eqns.
- Notion of the *quasiparticles* is applicable at late times
- Full treatment of *Leptogenesis*?